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Chebyshev Approximation by Polynomials Plus $B\phi(Cx)$

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Let $0 \leqslant \alpha < \beta < \infty$ and let $C[\alpha, \beta]$ be the space of continuous functions on $[\alpha, \beta]$. For $g \in C[\alpha, \beta]$, define

$$\| g \| = \sup\{| g(x)|: \alpha \leqslant x \leqslant \beta\}.$$

Let n be a fixed *positive* integer. Let ϕ be a function continuous on the real line and define

$$F(A, x) = \sum_{k=1}^{n} a_k x^{k-1} + a_{n+1}\phi(a_{n+2}x).$$

The approximation problem is, given $f \in C[\alpha, \beta]$, to find a parameter A^* minimizing $\|f - F(A, \cdot)\|$. Such a parameter A^* is called best and $F(A^*, \cdot)$ is called a best approximation to f. We are interested in those ϕ for which an alternating theory exists, that is, a theory in which best approximations are characterized by alternation of their error curve.

This problem has already been studied in the case $n = 1$ [4] and in the case $n = 0$, $\phi(0) \neq 0$ [2].

It turns out that many functions ϕ which we would like to use are not continuous on the real line, or not strictly monotonic on the real line, but are so on an interval. We therefore broaden the problem. Let ϕ be continuous on (μ, ν) (the interval may be infinite). Let γ and δ be given such that if $x \in [\alpha, \beta]$, $\rho \in (\gamma, \delta)$, then $\rho x \in (\mu, \nu)$. Let P be the space of all $n + 2$-tuples $(a_1, ..., a_{n+2})$, $\gamma < a_{n+2} < \delta$; then $\{F(A, \cdot): A \in P\} \subset C[\alpha, \beta]$. The approximation problem is to choose $A^* \in P$ minimizing $\|f - F(A, \cdot)\|$ over $A \in P$.

Let H_m denote the class of polynomials of degree $\leqslant m$. In the alternating theory to be developed, the approximations are of two types, namely, elements of H_{n-1} and approximations $F(A, \cdot)$ which are not in H_{n-1}. If $a_{n+1} = 0$, the nonlinear term vanishes, and if $a_{n+2} = 0$, it is constant, hence approximations $F(A, \cdot)$ not in H_{n-1} have $a_{n+1}a_{n+2} \neq 0$.

THEOREM. *Let ϕ have a continuous $(n + 1)$st derivative on (μ, ν). Let $\phi^{(n)}$*

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not vanish on (μ, ν) *and* $x\phi^{(n+1)}/\phi^{(n)}$ *be strictly monotonic on* (μ, ν). $F(A, \cdot)$ *is best to* f *if and only if* $f - F(A, \cdot)$ *alternates* $d(A)$ *times, where* $d(A) = n + 1$ *if* $F(A, \cdot) \in H_{n-1}$ *and* $d(A) = n + 2$ *if* $F(A, \cdot) \notin H_{n-1}$. *A best approximation is unique.*

This is a consequence of the theory of Meinardus and Schwedt [5, 144 ff; 6, 310], Lemma 3, Lemma 4, and Lemma 5.

Chebyshev Sets

LEMMA 1. *Let* $\psi^{(n)}$ *have no zeros on* (α, β), *then* $\{1,..., x^{n-1}, \psi\}$ *is a Chebyshev set on* $[\alpha, \beta]$.

This is problem 8 in Cheney [1, 77] and Lemma 4 of [3].

LEMMA 2. *Let* $\psi^{(n)}$ *not vanish on* (α, β) *and* $x\psi^{(n+1)}/\psi^{(n)}$ *be strictly monotonic on* $[\alpha, \beta]$. *Then* $\{1, x,..., x^{n-1}, \psi^{(n)}, x\psi^{(n+1)}\}$ *is a Chebyshev set on* $[\alpha, \beta]$.

This is the corollary to Lemma 5 of [3].

The Tangent Space

$$\frac{\partial}{\partial a_k} F(A, x) = x^{k-1}, \qquad k = 1,..., n;$$

$$\frac{\partial}{\partial a_{n+1}} F(A, x) = \phi(a_{n+2}x), \qquad \frac{\partial}{\partial a_{n+2}} F(A, x) = a_{n+1}x\phi'(a_{n+2}x).$$

Define for given $A \in P$, $B \in E_{n+2}$,

$$D(A, B, x) = \sum_{k=1}^{n+2} b_k \frac{\partial}{\partial a_k} F(A, x), \qquad S(A) = \{D(A, B, \cdot): B \in E_{n+2}\}.$$

LEMMA 3. *Let* $\phi^{(n)}$ *not vanish on* (μ, ν) *and* $x\phi^{(n+1)}/\phi^{(n)}$ *be strictly monotonic on* (μ, ν). *For any approximation in* H_{n-1} *there is a parametrization* $F(A, \cdot)$ *such that* $S(A)$ *is a Haar subspace of dimension* $n + 1$. *For any approximation not in* H_{n-1}, $S(A)$ *is a Haar subspace of dimension* $n + 2$.

Proof. We can express an element of H_{n-1} as

$$F(A, x) = a_1 + \cdots + a_n x^{n-1} + 0 \cdot \phi(a_{n+2}x), \qquad a_{n+2} \in (\gamma, \delta) \sim \{0\}.$$

in which case $S(A)$ is the linear space with basis $\{1,..., x^{n-1}, \phi(a_{n+2}x)\}$. As $\phi^{(n)}$ does not vanish on (μ, ν), $S(A)$ is a Haar subspace of dimension $n + 1$ by Lemma 1. Let $F(A, \cdot) \notin H_{n-1}$ then $a_{n+1}a_{n+2} \neq 0$ and $S(A)$ is the linear space with basis $\{1, \ldots, x^{n-1}, \phi(a_{n+2}x), x\phi'(a_{n+2}x)\}$. By Lemma 2, $S(A)$ is a Haar subspace of dimension $n + 2$.

Property Z

DEFINITION. F has *property Z* of degree m at A if $F(A, \cdot) - F(B, \cdot)$ having m zeros implies $F(A, \cdot) \equiv F(B, \cdot)$.

LEMMA 4. *Let $\phi^{(n)}$ have no zeros in (μ, ν), then F has property Z of degree $n + 1$ at all parameters corresponding to elements of H_{n-1}.*

Proof. The difference of an approximation in H_{n-1} and another approximation is of the form

$$a_1 + \cdots + a_n x^{n-1} + a_{n+1}\phi(a_{n+2}x), \qquad a_{n+2} \neq 0.$$

If $\phi^{(n)}$ has no zeros in (μ, ν), this has at most n zeros in (μ, ν) by Lemma 1.

LEMMA 5. *Let $\phi^{(n)}$ have no zeros in (μ, ν) and $x\phi^{(n+1)}/\phi^{(n)}$ be strictly monotonic on (μ, ν). Let $F(A, \cdot)$ not be in H_{n-1}, then F has property Z of degree $n + 2$ at A.*

Proof. Suppose $F(A, \cdot) - F(B, \cdot)$ has $n + 2$ zeros on $[\alpha, \beta]$. Suppose first that $a_{n+2} = b_{n+2} = \sigma \neq 0$, then

$$F(A, x) - F(B, x) = (a_1 - b_1) + \cdots + (a_n - b_n)\, x^n + (a_{n+1} - b_{n+1})\, \phi(\sigma x).$$

As $\phi^{(n)}(\sigma x)$ does not vanish on (α, β), this can have at most n zeros on $[\alpha, \beta]$ unless it vanishes identically by Lemma 1. Next we suppose that $a_{n+2} \neq b_{n+2}$. We have

$$F^{(n)}(A, x) - F^{(n)}(B, x) = a_{n+1}a_{n+2}^n\phi^{(n)}(a_{n+2}x) - b_{n+1}b_{n+2}^n\phi^{(n)}(b_{n+2}x),$$

and this must have at least 2 zeros in (α, β). $F(B, \cdot) \notin H_{n-1}$ by the previous lemma so $b_{n+1}b_{n+2} \neq 0$, and we have

$$\frac{a_{n+1}a_{n+2}^n}{b_{n+1}b_{n+2}^n} = \frac{\phi^{(n)}(b_{n+2}x)}{\phi^{(n)}(a_{n+2}x)}$$

at a zero x of $F^{(n)}(A, \cdot) - F^{(n)}(B, \cdot)$. As $\phi^{(n)}(b_{n+2}x)/\phi^{(n)}(a_{n+2}x)$ takes the same value at the two points, its derivative

$$\frac{b_{n+2}\phi^{(n+1)}(b_{n+2}x)\,\phi^{(n)}(a_{n+2}x) - a_{n+2}\phi^{(n+1)}(a_{n+2}x)\,\phi^{(n)}(b_{n+2}x)}{[\phi^{(n)}(a_{n+2}x)]^2} \tag{1}$$

has a zero z in (α, β). But if the numerator of (1) vanishes at z, we have

$$\frac{b_{n+2}z\phi^{(n+1)}(b_{n+2}z)}{\phi^{(n)}(b_{n+2}z)} = \frac{a_{n+2}z\phi^{(n+1)}(a_{n+2}z)}{\phi^{(n)}(a_{n+2}z)} \tag{2}$$

which contradicts strict monotonicity of $x\phi^{(n+1)}/\phi^{(n)}$ on (μ, ν), proving the lemma. If $[\alpha, \beta]$ contained zero as an interior point (2) could hold with $z = 0$, hence the proof would not go through. It should also be noted that the proof of the lemma guarantees that $F(A, \cdot) \notin H_{n-1}$ has a *unique* representation.

Examples

EXAMPLE 1. Let $\phi(x) = \exp(x)$ and $(\mu, \nu) = (\gamma, \delta) = (-\infty, \infty)$.

$$\phi^{(n)}(x) = \exp(x), \qquad x\phi^{(n+1)}(x)/\phi^{(n)}(x) = x$$

EXAMPLE 2. Let $\phi(x) = \log(1 + x)$ and $(\mu, \nu) = (-1, \infty)$.

$$\phi^{(n)}(x) = (-1)^{n+1}\, n!/(1 + x)^n$$

$$x\phi^{(n+1)}(x)/\phi^{(n)}(x) = -nx/(1 + x) = n\left[\frac{1}{1 + x} - 1\right]$$

EXAMPLE 3. Let $\phi(x) = 1/(1 + x)$ and $(\mu, \nu) = (-1, \infty)$. We note that this ϕ is the derivative of the ϕ of the previous example.

Even and Odd ϕ

$$\begin{aligned} a_{n+1}\phi(a_{n+2}x) &= a_{n+1}\phi(-a_{n+2}x) && \phi \text{ even} \\ &= -a_{n+1}\phi(-a_{n+2}x) && \phi \text{ odd,} \end{aligned}$$

hence, if ϕ is even or odd, we need consider only parameters A with $a_{n+2} \geqslant 0$. We obtain by arguments similar to the preceding

THEOREM. *Let ϕ be even or odd. Let ϕ have a continuous second derivative on $(-\nu, \nu)$. Let $\phi^{(n)}$ not vanish on $(0, \nu)$ and $x\phi^{(n+1)}/\phi^{(n)}$ be strictly monotonic on $[0, \nu)$. $F(A, \cdot)$ is best to f if and only if $f - F(A, \cdot)$ alternates $d(A)$ times, where $d(A) = n + 1$ if $F(A, \cdot) \in H_{n-1}$ and $d(A) = n + 2$ if $F(A, \cdot) \notin H_{n-1}$. Best approximations are unique.*

Even and Odd Examples

We note that the derivatives of sine, cosine, hyperbolic sine, and hyperbolic cosine repeat themselves. From this observation, and the examples for the case $n = 1$ [4], it is easily seen that for all $n \geq 1$ the preceding theorem holds for sine and cosine with $\nu = \pi/2$ and for hyperbolic sine and cosine with $\nu = \infty$.

Discrete Approximation

An alternating theory for approximation on an interval implies an alternating theory for approximation by the same family on a finite subset of the interval. For a precise statement and proof of a more general result, see [7].

Applications

The major applications of the theory is probably to Chebyshev fitting of data on finite sets. Two examples where an approximation of the form studied in this paper is desired are Stewart [9], in which case the approximation is of the form

$$F(A, x) = a_1 + a_2 \log(1 + a_3 x),$$

and Heitkamp, Merwitz, and Spatz [8, 408], in which case the approximation is of the form

$$F(A, x) = a_1 + a_2 x + a_3 \exp(a_4 x).$$

Shah and Khatri [10] consider approximation by $\beta\rho^x$ plus a constant or first degree polynomial, and cite numerous applications. Writing $\rho^x = e^{(\log \rho)x}$ converts their form into our form with $\phi = \exp$.

Computation of Best Approximations

The alternation result suggests use of a variant of the Remez algorithm. Some discussion of the case $n = 1$ appears in [4] and it is not hard to see how it can be generalized to other n. The author has written and run programs to compute best approximations on an interval and finite set by the Remez algorithm.

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